The Strength of Indirect Relations in Social Networks

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1. Introduction
2. Indirect Relations
3. Strong–Weak Direct–Indirect Relations
4. Further Directions – Empirical Data

Transitive completion

- Triadic closure:
  - Georg Simmel (1858–1918) (Triad in Conflict and the Web of Group Affiliations, 1922 [1955])
  - Fritz Heider (1896–1988) (Balance Theory, 1946)
  - Harrison C. White (Notes on the constituents of social structure, 1965 [2008])
  - Ronald Breiger (The duality of persons and groups, 1974)
  - Vast literature on inter–locking directorates, co–citation analysis, social movements and participation studies etc.
  - Duncan Watt & Steven Strogatz (Clustering coefficient, 1998)
- Polygonal closures (quadruple, quintuple etc.):
  - V. Batagelj & M. Zaveršnik (Short cycle connectivity, 2007)
  - Emmanuel Lazega and co–workers (Multi-level network analysis through linked design, 2008, Network parachutes from tetradic substructures, 2010, etc.)

Engineering Networks

- Agglomerative methods:
  - Preferential attachment or Matthew effect
  - Cntangion, cascades, percolation
  - The third man argument (Plato) or Bradley's regress
- Divisive methods:
  - Structural partitions:
    - Hierarchical clustering
    - Blockmodeling, equivalence classes
    - Cliques, components, core–periphery etc.
    - Community structure
  - Categorizational partitions:
    - Attributes (homophily)
    - Attitudes (signed networks, balance theory)
  - Multi–Level (typological) partitions:
    - Individual–collective actors
    - Geographical networks etc.

Dual Graph System

- A **bipartite graph** $H(U, V) = (U, V, E)$ with vertex classes U and V ($U \cap V = \varnothing$) and E a set of connections (or associations or "translations") between U and V, i.e., $E \subset U \times V$.[1] Let A denote the adjacency matrix of $H(U, V)$.
- A (simple undirected) **graph** $G(U) = (U, E_U)$ on the set of vertices U and with a set of edges $E_U \subset U \times U$, for which A_U is its adjacency matrix.
- A (simple undirected) **graph** $G(V) = (V, E_V)$ on the set of vertices V and with a set of edges $E_V \subset V \times V$, for which A_V is its adjacency matrix.
- $\boxed{\text{Dual Graph System: } G = (U \cup V, E_U \cup E \cup E_V)}$

[1] By considering V as a collection of subsets of U (i.e., V as a subset of $\mathcal{P}(U)$, the power set of U, that is the set of all subsets of U), the bipartite graph $H(U, V)$ is the *incidence graph* that corresponds (in a 1–1 way) to the *hypergraph* $H = (U, V)$ (Bollobas, 1998, p. 7).

An Example of a Dual Graph System

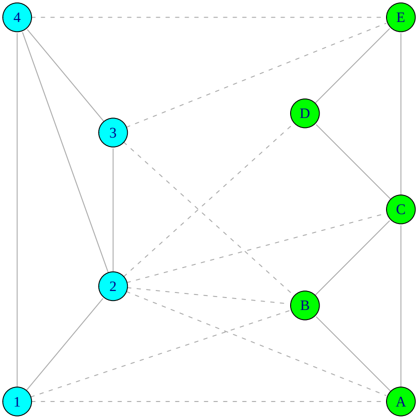

Figure: A dual graph system composed of two graphs $G(U)$ and $G(V)$, which are "translated" to each other by a bipartite graph $H(U, V)$ (with dashed edges), where $U = \{1, 2, 3, 4\}$ and $V = \{A, B, C, D, E\}$.

A Vertex–Attributed Graph as a Dual Graph System

- Let $G_\alpha(W) = (W, F)$ be a graph with set of vertices W and set of edges $F \subset W \times W$.
- Let all vertices be equipped with an **attribute**, defined by the assignment mapping $\alpha\colon W \to \{0, 1\}$, such that, for any vertex $w \in W$, $\alpha(w) = 1$, when the vertex satisfies the attribute, and $\alpha(w) = 0$, otherwise.
- Setting:
  - $U = \{w \in W\colon \text{such that } \alpha(w) = 1\}$,
  - $V = \{w \in W\colon \text{such that } \alpha(w) = 0\}$,
  - $E_U = \{(w_p, w_q) \in W \times W\colon \text{ such that } \alpha(w_p) = \alpha(w_q) = 1\}$,
  - $E_V = \{(w_r, w_s) \in W \times W\colon \text{ such that } \alpha(w_r) = \alpha(w_s) = 0\}$,
  - $E = \{(w_p, w_r) \in W \times W\colon \text{such that } \alpha(w_p) = 1 \text{ and } \alpha(w_r) = 0\}$.
- Then $G_\alpha(W)$ becomes a dual graph system $G_\alpha = (U \cup V, E_U \cup E \cup E_V)$.

Paths

- Let $G = (W, F)$ (undirected) graph.
- A **path of length** n (or n-**path**) in G, from a_1 to a_n, is formed by a sequence of vertices $a_1, a_2, \ldots, a_n \in W$ such that $(a_j, a_{j+1}) \in F$, for all $j = 1, 2, \ldots, n-1$, where all vertices are distinct (except possibly the 2 terminal ones).
- A n-path from a_1 to a_n is denoted as $(a_1, \ldots, a_n)$.
  - If $a_1 \neq a_n$, the path $(a_1, \ldots, a_n)$ is *open*.
  - If $a_1 = a_n$, the path $(a_1, \ldots, a_{n-1}, a_1)$ is *closed* and it forms a $(n-1)$-*cycle*.
  - For $n = 0$, a 0-path reduces to a vertex.
- The (transitive) *closure* of a path $(a_1, \ldots, a_n)$, denoted as $\overline{(a_1, \ldots, a_n)}$, is defined as follows:

$$\overline{(a_1, \ldots, a_n)} = \begin{cases} (a_1, a_n), & \text{when } n \geq 1 \text{ and } a_1 \neq a_n, \\ \{a_0\}, & \text{when } n = 0 \text{ and } a_1 = a_n = a_0. \end{cases}$$

A Signed Graph as a Dual Graph System

- Let $G(U) = (U, E_U)$ be a graph.
- Let $G(V) = (\{p, q\}, \{(p, q)\})$ be a *dipole*.
- Suppose that there exist "translations" from *all* vertices of U to V, i.e., $E = \{(u, p) \cup (u, q)\text{: for all } u \in U\}$.
- Define the *sign* of each edge in $G(U)$ by an assignment mapping σ: $E_U \to \{+, -\}$ as follows, for any $(u_i, u_j) \in E_U$:
    - $\sigma(u_i, u_j) = +$, whenever both u_i and u_j are "translated" to the same pole, and
    - $\sigma(u_i, u_j) = -$, otherwise.
- Then, for all $(u_i, u_j) \in U$,
    - $\sigma(u_i, u_j) = +$ if and only if $(u_i, u_j) = \overline{(u_i, p, u_j)} = \overline{(u_i, q, u_j)}$ and
    - $\sigma(u_i, u_j) = -$ if and only if $(u_i, u_j) = \overline{(u_i, p, q, u_j)} = \overline{(u_i, q, p, u_j)}$.

Polarity

- If $G = (U \cup V, E_U \cup E \cup E_V)$ is a dual graph system with $E \neq \varnothing$, we denote, for all $u \in U$ and $v \in V$:
    - the *right polar set* of u as $\{u\}'$ to be the set of those $v \in V$, which are all "translated" to u through the existing traversal bridges (i.e., such that the uv–entry of A is equal to 1),
    - Similarly, the *left polar set* of v as $'\{v\}$ to be the set of those $u \in U$ such that $v \in \{u\}'$.
- Thus, denoting by $|X|$ the cardinality of a set X, the *traversal degrees* of a vertex $u \in U$ or a vertex $v \in V$ are defined (respectively) as:

$$\begin{aligned} \deg_{UV}(u) &= |\{u\}'| = |\{(u,v) \in E \colon v \in V\}| = \sum_{v \in V} A_{uv}, \\ \deg_{UV}(v) &= |'\{v\}| = |\{(u,v) \in E \colon u \in U\}| = \sum_{u \in U} A_{uv}. \end{aligned}$$

General Definition of Indirect Relations

- **Edges = Direct Relations**
- **Closures of (n-)Paths = (n-th Order) Indirect Relations**

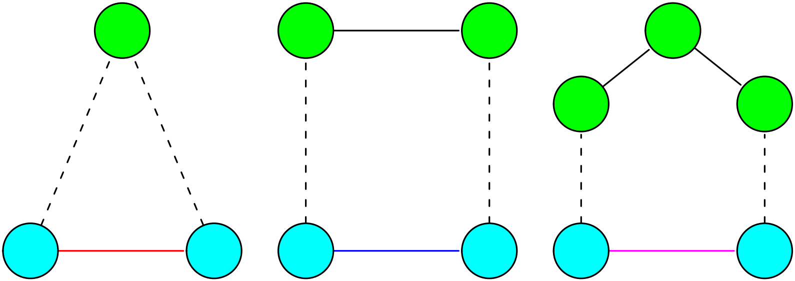

Figure: The direct relations, on the top, are the black colored continuous lines or, in the middle, the dashed lines (traversal relations), while the induced indirect relations are, at the bottom, colored as follows: 0-th order red, 1-st order blue and 2-nd order magenta.

General Notation for Indirect Relations

- Let $G = (W, F)$ be a graph.
- $E_n^\star(W)$ is the set of all n-th order indirect relations in W.
- $G_n^\star(W) = (W, E_n^\star(W))$ is the corresponding graph.
- If $(w_i, w_j) \in E_n^\star(W)$, the *weight* $\omega_n(w_i, w_j)$ is equal to the total number of existing indirect relations on (w_i, w_j), where $\omega_n(w_i, w_j) = 0$, whenever $(w_i, w_j) \notin E_n^\star(W)$.
- $A_W^{(n)\star} = \{\omega_n(w_i, w_j)\}$ is the adjacency matrix of the (weighted) graph $G_n^\star(W)$.
- The *minimal order weight* is $\omega_{\nu_{i,j}}(w_i, w_j)$, where $\nu_{i,j}$ is the minimum of all appropriate integers n, for which $(w_i, w_j) \in E_n^\star(W)$.
- $G^\star(W)$ is the minimal order weighted graph and
- $A_W^\star = \{\omega_{\nu_{i,j}}(w_i, w_j)\}$ is the corresponding adjacency matrix.

Indirect Relations in Graphs

- Let $G(W) = (W, F)$ a graph with adjacency matrix A.
- For any integer $n \geq 2$ and for any two (distinct) vertices $w_p, w_q \in W$, there exists an induced ***n*-th order indirect relation** between w_p and w_q, $(w_p, w_q) \in E_n^\star(W)$, if there exists a n-path $(w_p, w_{p+1}, \ldots, w_{p+n-1}, w_q)$ in $G(W)$ such that $\overline{(w_p, w_{p+1}, \ldots, w_{p+n-1}, w_q)} = (w_p, w_q)$.
  - 2-nd order indirect relations generated by a **triadic closure**.
  - 3-rd order indirect relation generated by a **quadruple closure**.
  - And so on, for any **"polygonal" closure**, up to $\mathrm{diam}(W) - 1$.
- However, the adjacency matrices $A_W^{(n)\star}$ of graphs $G_n^\star(W)$ *cannot* be computed by powers of A (walks *not* paths).
- This is the problem of "self-avoiding walks" (Hayes, 1998).
- Remarkably, Leslie G. Valiant (1979) has shown that this problem is *#P-complete* under polynomial parsimonious reductions (for any directed or undirected graph).

Indirect Relations in Bipartite Graphs

- Let $H(U, V) = (U, V, E)$ a bipartite graph with adjacency matrix A.
- Since now any k-path, for $k \geq 2$, is composed of successively alternating "translations" from U to V and from V to U (or vice versa), two (distinct) vertices in the same class of vertices (mode) can be connected by a k-path only as far as k is even.
- Thus, for any integer $2n \geq 2$ and for any two distinct vertices $u_p, u_q \in U$, there exists an induced $2n$-**th order indirect relation** between u_p and u_q, $(u_p, u_q) \in E^{\star}_{2n}(U)$, if there exist two vertices $v_r, v_s \in V$ and a $(2n-2)$-path $(v_r, u_r, v_{r+1}, \ldots, v_{r+n-2}, u_{r+n-2}, v_s)$ in $H(U, V)$ such that
    - u_p is "translated" to v_r and u_q to v_s and
    - $\overline{(u_p, v_r, u_r, v_{r+1}, \ldots, v_{r+n-1}, u_{r+n-1}, v_s, u_q)} = (u_p, u_q)$.
- Similarly (by duality), one defines an induced $2n$-th order indirect relation in V, $(v_r, v_s) \in E^{\star}_{2n}(V)$.

Indirect Relations in Bipartite Graphs (Cont.)

- Any 2-nd order indirect relation is generated by the formal mechanism of **triadic closure** in $G(U, V)$.
  - The graphs $G_2^\star(U)$ & $G_2^\star(V)$ coincide with the 2 projections of the bipartite graph $H(U, V)$ onto U and V, respectively.
  - Thus, the corresponding adjacency matrices are:
    - $A_U^{(2)\star} = AA^T$ and
    - $A_V^{(2)\star} = A^T A$, respectively.
- However, again, for any $n \geq 2$, the adjacency matrices $A_W^{(2n)\star}$ of graphs $G_{2n}^\star(W)$ *cannot* be computed by powers of A (walks *not* paths).

Indirect Relations in Dual Graph Systems

- Let $G = (U \cup V, E_U \cup E \cup E_V)$ be a dual graph system.
- The *most general definition of induced indirect relations* is the following (formulated on U and, similarly, by duality, on V):
- For any three integers $m, n, k \geq 0$ and for any two vertices $u_p, u_q \in U$, we will say that there exists a (m, n, k)**-th order indirect relation** between u_p and u_q, $(u_p, u_q) \in E^\star_{(m,n,k)}(U)$, if there exist the following vertices and paths:
  - a vertex $u_i \in U$ and a path $(u_p, u_{p+1}, \ldots, u_{p+m-1}, u_i)$ of length m from u_p to u_i in $G(U)$,
  - a vertex $u_j \in U$ and a path $(u_q, u_{q+1}, \ldots, u_{q+k-1}, u_j)$ of length k from u_q to u_j in $G(U)$ and
  - two vertices $v_r, v_s \in V$ and a path $(v_r, v_{r+1}, \ldots, v_{r+n-1}, v_s)$ of length n from v_r to v_s in $G(V)$,
  - such that
    - u_i "translates" traversally (from $G(U)$ to $G(V)$) to v_r and u_j "translates" traversally to v_s and
    - $\overline{(u_p, \ldots, u_i, v_r, \ldots, v_s, u_j, \ldots, u_q)} = (u_p, u_q)$.

Indirect Relations in Dual Graph Systems (Cont.)

- A *more particular definition of induced indirect relations* in the dual graph system $G = (U \cup V, E_U \cup E \cup E_V)$ (to follow in the sequel) is for $m = k = 0$:
- For any integer $n \geq 0$ and for any two distinct vertices $u_p, u_q \in U$, there exists an induced ***n*-th order indirect relation** between u_p and u_q, $(u_p, u_q) \in E_n^\star(U)$, if there exist two vertices $v_r, v_s \in V$ and a n-path $(v_r, v_{r+1}, \ldots, v_{r+n-1}, v_s)$ in $H(V)$ such that
  - u_p is "translated" to v_r and u_q to v_s and
  - $\overline{(u_p, v_r, v_{r+1}, \ldots, v_{r+n-1}, v_s, u_q)} = (u_p, u_q)$.
- Similarly (by duality), one defines an induced n-th order indirect relation between v_r and v_s in V, $(v_r, v_s) \in E_n^\star(V)$.

Indirect Relations in Dual Graph Systems (Cont.)

- Any 0-th order indirect relation is generated by the formal mechanism of **triadic closure** in the dual graph system G.
  - The graphs $G_0^\star(U)$ & $G_0^\star(V)$ coincide with the 2 projections of the bipartite graph $H(U, V)$ onto U and V, respectively.
  - Thus, the corresponding adjacency matrices are:
    - $A_U^{(0)\star} = AA^T$ and
    - $A_V^{(0)\star} = A^T A$, respectively.
- Similarly, any 2-nd order indirect relation is generated by the formal mechanism of **quadruple closure** in the dual graph system G.
  - The adjacency matrices of graphs $G_2^\star(U)$ & $G_2^\star(V)$ are:
    - $A_U^{(2)\star} = A(A_V)A^T$ and
    - $A_V^{(2)\star} = A^T(A_U)A$, respectively.
- However, no further analytic computation is possible for the adjacency matrices $A_W^{(n)\star}$ of graphs $G_n^\star(W)$, when $n \geq 3$, for the same reason as the aforementioned above.

Example

The following are the adjacency matrices of the example:

$$A_U = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, A_U^{(0)\star} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A_U^{(1)\star} = \begin{bmatrix} 0 & 4 & 1 & 0 \\ 4 & 0 & 4 & 2 \\ 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix},$$

$$A_U^{(2)\star} = \begin{bmatrix} 0 & 6 & 3 & 2 \\ 6 & 0 & 7 & 4 \\ 3 & 7 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{bmatrix}, A_U^{(3)\star} = \begin{bmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 6 & 4 \\ 4 & 6 & 0 & 2 \\ 4 & 4 & 2 & 0 \end{bmatrix}, A_U^{(4)\star} = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix},$$

$$A_V = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, A_V^{(0)\star} = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, A_V^{(1)\star} = \begin{bmatrix} 0 & 3 & 1 & 1 & 3 \\ 3 & 0 & 2 & 2 & 4 \\ 1 & 2 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 2 \\ 3 & 4 & 2 & 2 & 0 \end{bmatrix},$$

$$A_V^{(2)\star} = \begin{bmatrix} 0 & 5 & 1 & 1 & 6 \\ 5 & 0 & 2 & 2 & 7 \\ 1 & 2 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 6 & 7 & 3 & 3 & 0 \end{bmatrix}, A_V^{(3)\star} = \begin{bmatrix} 0 & 5 & 1 & 1 & 4 \\ 5 & 0 & 2 & 2 & 5 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 4 & 5 & 1 & 1 & 0 \end{bmatrix}.$$

Thus, the adjacency matrices of the graphs equipped with the *minimal order* indirect relations are:

$$A_U^\star = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}, A_V^\star = \begin{bmatrix} 0 & 2 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 \\ 3 & 1 & 2 & 2 & 0 \end{bmatrix}.$$

Example (Cont.)

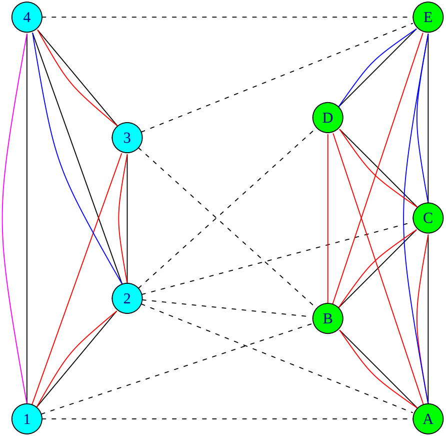

Figure: Direct relations are colored black (with dashed traversals), while minimal order indirect relations are colored as follows: 0-th order red, 1-st order blue and 2-nd order magenta.

Consequencies and Terminology

- A vertex is incident to an indirect relation only as far as it "translates" between the dual graphs.
- After Harrison C. White, a direct relation is said to **institutionalize** an indirect relation, if the latter forms at the same directly linked edge with the former.
- A direct relation is called **detached** if it does not institutionalize any indirect relation. Any edge incident to an "untranslated" vertex is always detached.
- An indirect relation is called **emergent** if it is not institutionalized (i.e., it forms on an edge, which is not directly linked).

Consequencies and Terminology (Cont.)

- Every indirect relation is of order 0 if and only if, ignoring vertices with traversal degree ≤ 1, the dual graph system is filled up completely by triangles, which have two edges as "translations," while the third one is either an institutionalizing direct relation or an emergent indirect relation. In particular, every indirect relation is 0-th order emergent if and only if the dual graph system is bipartite.
- Every indirect relation is of order 1 if and only if, ignoring vertices with traversal degree $= 0$, the dual graph system is filled up completely with rectangles, which have two parallel edges as "translations," while the other are either two institutionalizing direct relations or a detached direct relation and an emergent indirect relation. In particular, every indirect relation is 1-st order institutionalized if and only if the dual graph system constitutes a graph isomorphism and "translations" are just permutations of the same (common) vertex set.

Recontextualizing Granovetter's Weak–Strong Ties

- **Mark Granovetter (1973)**: "*The degree of overlap of two individual's friendship networks varies directly with the strength of their tie to one another.*"
- However, although Granovetter was considering a single network, where all actors were embedded, he had to distinguish between two types of ties (strong or weak) among actors in this social network.
- *Our point*: If one manages to dispense with the single common network assumption and, in its place, one considers circumspectly a dual network system, then the two dual networks might be used as a "leverage" in order to facilitate a clearer formal analysis of the distinction between strong–weak ties (for instance, compensating for any definitional ambiguity on issues of measurement).

Weak or Strong Direct or Indirect Relations

- Granovetter defined network strength in terms of frequency or duration (time), intensity (closeness), intimacy (self–determination) and reciprocity (trust) of ties.
- Direct relations are the basis of ties among persons and as such they might instantiate:
  - Either strong ties (as in an ordinary matrimonial relationship),
  - Or weak ties (as in a frustrated intimate relationship).
- Indirect relations, as the anthropologist Siegfried Frederick Nadel has argued that "membership roles" correspond completely to "relational roles", might instantiate:
  - Either strong ties (as in some online friendships),
  - Or weak ties (as in an acquaintanceship with very low degree of overlap).

In What Sense Can Network Duality Lever the Measurement of Tie Strength?

- A possible answer could be based on the requirement of having the micro and macro levels of sociological theory linked together (that was exactly and explicitly Granovetter's motivation). For instance, typically, in a dual network system:
  - The first network concerns the micro–interactions of actors in their friendship network.
  - And the dual network concerns the macro consequences or the emerging morphogenetic patterns, which supervene these interactions, or the macro settings, categorizations (distinctions, partitions etc.) or any existing (enabling or constricting) framing that may influence, reshuffle or rearticulate the structure and the dynamics of these interactions.

Examples of Micro–Macro Dual Social Networks

- Possible examples of ramifications of such scenarios:
  - *On institutional dimensions*: The dual of the friendship network might be taken to be a network of groups, circles, clubs or organizations (or any other group categorizations), to which individuals are affiliated in their friendship micro–interactions.
  - *On cultural dimensions*: The dual might be a network of events, distinct tastes, preferences, attitudes or polarized values (or any other habitual significations), with which individuals are engaged in their friendship micro–interactions.
- In any case, it would be hard to imagine that there exists a social network (say, of micro interactions), which would be absolutely isolated, abruptly cut from any traversal or border crossing "translations" to another dual social network (say, of corresponding implications, constraints or opportunities, at the macro level).

Some Notation

- Let $G = (U \cup V, E_U \cup E \cup E_V)$ be a dual graph system.
- Let us focus on the graph $G(U) = (U, E_U)$ and define the following sets of relations:
  - $\mathcal{R}_{ID} = E_U \cap E_n^\star(U)$ (for some n), i.e., $\mathcal{R}_{ID}$ is the set of all institutionalizing direct relations in G.
  - $\mathcal{R}_{EI} = E_n^\star(U) \setminus E_U$ (for some n), i.e., $\mathcal{R}_{EI}$ is the set of all emergent indirect relations in G.
- As far as an arbitrary pair of vertices $(u_i, u_j) \in U \times U$ does happen to form an indirect relation, it has to be one of the above two types.

Measuring the Strength of Direct and Indirect Relations through a Utility Function

Let δ: $\mathcal{R}_{ID} \cup \mathcal{R}_{EI} \to \mathbb{R}^+$ be a **utility function** defined as follows:

- For any $(u_i, u_j) \in \mathcal{R}_{ID}$,

$$\delta(u_i, u_j) = \frac{c}{1 + \nu_{i,j}},$$

  where c is a normalization constant and $\nu_{i,j}$ is the minimal order of all indirect relations institutionalized by (u_i, u_j).

- For any $(u_i, u_j) \in \mathcal{R}_{EI}$,

$$\delta(u_i, u_j) = \frac{1}{1 + n_{i,j}},$$

  where $n_{i,j}$ is the order of the emergent indirect relation (u_i, u_j).

Ordering Relations with the Utility Function

For $(u_i, u_j), (u_k, u_l)$ two relations in $\mathcal{R}_{ID} \cup \mathcal{R}_{EI}$:

- We say that (u_i, u_j) is **stronger** than (u_k, u_l) (or that (u_k, u_l) is **weaker** than (u_i, u_j)) whenever

$$\delta(u_i, u_j) > \delta(u_k, u_l).$$

- If $\delta(u_i, u_j) = \delta(u_k, u_l)$, (u_i, u_j) is stronger than (u_k, u_l) (or (u_k, u_l) is weaker than (u_i, u_j)) whenever

$$\omega_\nu(u_i, u_j) > \omega_\nu(u_k, u_l),$$

  where ω_ν is the *weight* of either the corresponding minimal order relation (denoted as ω_μ), when comparing two institutionalizing direct relations, or of the emergent relation, when comparing two emergent relations.

- The constant c is chosen in such a way that any institutionalizing direct relation is *a priori* stronger than any emergent indirect relation.

Further Directions – Empirical Data

For empirical data of concrete dual social network systems, we intend to test statistically the following Hypotheses:

Hypothesis 1 Is the closure of a *n*-path of direct relations a strong indirect relation, which is emergent or institutionalized by a direct relation?

Hypothesis 2 If a bridge is formed by an institutionalizing direct relation or an emergent indirect relation, then are the latter weak relations?

A final interesting Hypothesis to test is related to how "structurally balanced" a community partitioning could be:

Hypothesis 3 Do traversal relations among different communities tend to be weak emergent indirect relations and internal relations inside communities tend to be strong direct relations?